Last update: 4:46pm, Feb 3.

Math 565: Functional Analysis | HOMEWORK 2 | **Due: Feb 3, 23:59**

1. Let M be a proper closed subspace of a normed vector space X and let $\pi : X \to X/M$ denote the projection map $x \mapsto x + M$. Note that π is a surjective linear transformation and prove that $\|\pi\| = 1$.

2. Let X be a normed vector space and $T : X \to \mathbb{C}^d$ be a linear transformation (e.g. T is a linear functional). Prove that T is continuous (equivalently, bounded) if and only if $T^{-1}(0)$ is closed in X.

HINT: First isomorphism theorem.

3. Prove that every infinite-dimensional normed vector space X contains a sequence $\{x_n\}$ in X of normal vectors (i.e. $\|x_n\| = 1$) whose pairwise distance is at least $1/2$. In particular, X is not locally compact.

HINT: Construct the sequence inductively.

4. Let X be a Banach space.

(a) Show that if Y, Z are Banach spaces, and $S \in B(X, Y)$, $T \in B(Y, Z)$,

$$\|TS\| \leqslant \|T\| \, \|S\|,$$

where $TS \coloneqq T \circ S$. Thus $B(X) \coloneqq B(X, X)$ is a **Banach algebra**.

(b) Prove that if $T \in B(X)$ and $\|T\| < 1$, then $(1 - T)$ is invertible and $(1 - T)^{-1} \in B(X)$[1].

HINT: Analogy with the scalar identity $(1 - \alpha)^{-1} = \sum_{n \in \mathbb{N}} \alpha^n$ for $0 \leqslant \alpha < 1$.

(c) Deduce that if $T \in B(X)$ has an inverse in $B(X)$, then every $S \in B(X)$ with

$$\|T - S\| < \|T^{-1}\|^{-1},$$

is invertible and $S^{-1} \in B(X)$.

(d) Conclude that the set of invertible elements in $B(X)$ is open.

Definition. A Hausdorff topological space X is called **locally compact** if every point in X admits a compact neighbourhood. We abbreviate locally compact Hausdorff as **lcH**.

5. LcH spaces. Learn *all* of the following facts and prove *one* of them.

(a) Let X be Hausdorff topological space. Then any two disjoint compact subsets K_1, K_2 are separable by open sets, i.e. there are open $U_i \supseteq K_i$, $i = 1, 2$, such that U_1 and U_2 are disjoint. In particular, compact Hausdorff spaces are **normal**, i.e. any two disjoint closed sets C_1, C_2 are separated by disjoint open sets $U_1 \supseteq C_1$ and $U_2 \supseteq C_2$.

[1] By the open mapping theorem (which we will cover later), for a Banach space X, whenever $L \in B(X)$ is invertible, its inverse L^{-1} is automatically in $B(X)$.

(b) Let X be an lcH space. Let $K \subseteq U \subseteq X$, where K is compact and U is open. Then there is an open V such that $K \subseteq V \subseteq \overline{V} \subseteq U$ and $\overline{V}$ is compact.

HINT: First prove that there is an open U' such that $K \subseteq U' \subseteq U$ such that $\overline{U'}$ is compact, in particular, $\partial U'$ is compact.

(c) **Urysohn lemma for lcH.** Let X be an lcH space and $K \subseteq U \subseteq X$, where K is compact and U is open. There there is an $f \in C_c(X)$ such that $\mathbb{1}_K \leqslant f \leqslant \mathbb{1}_U$. You may use without proof the original Urysohn lemma, which says that in a normal topological space, for any two disjoint closed sets C_0, C_1, there is a continuous function f such that $f|_{C_i} = i$ for $i = 0, 1$.

(d) Let X be a second countable lcH space. Prove that every open set $U \subseteq X$ is K_σ, i.e. is a countable union of compact sets.

(e) **Partitions of unity.** Let X be a lcH space and $K \subseteq X$ a compact set. Prove that every finite open cover $\{U_i\}_{i<n}$ of K admits a **subordinate partition of unity on** K, namely, compactly supported functions $\{f_i\} \subseteq C_c(X, [0, 1])$ such that

$$\sum_{i<n} f_i|_K = 1 \quad \text{and} \quad \operatorname{supp} f_i \subseteq U_i \text{ for each } i < n.$$

Definition. Let X be a Hausdorff topological space. A **Radon measure** on X is a Borel measure μ on X which is finite on compact sets, outer regular on all Borel sets $B \subseteq X$, i.e.

$$\mu(B) = \inf \{\mu(U) : U \supseteq B \text{ open}\},$$

and tight on all open sets $U \subseteq X$, i.e.

$$\mu(U) = \sup \{\mu(K) : K \subseteq B \text{ compact}\}.$$

We proved in Lecture 10 of Math 564 that for lcH spaces, finiteness on compact sets is equivalent to local finiteness (that is: every point admits an neighbourhood of finite measure).

6. (a) Prove that on every topological space X, every Radon measure on X is tight on all Borel sets of finite measure.

HINT: For a Borel set $B \subseteq X$, take an open set $U \supseteq B$ approximating B, use tightness for U and outer regularity for $U \setminus B$.

(b) Prove that all finite Borel measures on second countable lcH spaces are Radon.

HINT: In Lectures 9–10 of Math 564, we proved that all finite Borel measures on metric spaces are outer and inner regular (approximated above by open and below by closed sets), where the only property of metric spaces we used is that open sets are F_σ (countable union of closed sets). For second countable lcH spaces an even stronger property holds: every open set is K_σ (Question 5(d)).

7. Let X be an lcH space and μ be a Radon measure on X. Prove that $C_c(X)$ is a dense subset of $L^p(X, \mu)$ (in the L^p norm), for all $1 \leqslant p < \infty$.

8. Let X be a normed vector space and $Y \subseteq X$ a dense subspace. Prove that every $I \in Y^*$ uniquely extends to a bounded linear functional $\tilde{I}$ on X; furthermore, $\|\tilde{I}\| = \|I\|$. Conclude that Y^* and X^* are isometrically isomorphic.

HINT: Lipschitz functions map Cauchy sequences to Cauchy sequences.

9. Positive linear functionals are compactly bounded. Let X be a topological space. Call a linear functional I on $C_c(X)$ **positive** if $If \geqslant 0$ for all $f \geqslant 0$ from $\mathbb{C}_c(X)$. For each compact $K \subseteq X$, note that

$$C_K(X) := \{f \in C_c(X) : \operatorname{supp} f \subseteq K\},$$

is a closed subspace of $C_c(K)$ in the uniform norm. Prove that if X is lcH and I is a positive linear functional on $C_c(X)$, then for each compact $K \subseteq X$, the restriction $I|_{C_K(X)}$ is a *bounded* linear functional on $C_K(X)$, i.e. there is a constant $\alpha_K > 0$ such that $|If| \leqslant \alpha_k \|f\|_u$ for all $f \in C_K(X)$.

10. [*Optional*] Let $(X, \mathcal{B})$ be a measurable space. Call measures μ and ν on $(X, \mathcal{B})$ **equivalent**, and write $\mu \sim \nu$, if $\mu \ll \nu$ and $\nu \ll \mu$. Now let X be a topological space and prove that the properties of regularity and tightness are invariant under $\sim$ on all finite Borel measures on X, i.e. if μ and ν are finite Borel measures with $\mu \sim \nu$, then μ is regular (resp. tight) if and only if ν is.

HINT: Recall the quantitative criterion for $\mu \ll \nu$ for finite measures, namely, that for each $\varepsilon > 0$ there is $\delta > 0$ such that for all $\mathcal{B}$-measurable sets $B \subseteq X$, $\mu(B) \leqslant \varepsilon$ whenever $\nu(B) \leqslant \delta$.